Name:

Derivatives: *Average Rate and Instantaneous Rate*

In Algebra I, you had problems like the following:

> It took Jack and Jill 30 minutes to walk one half mile. How fast did they walk?
> D = 0.5 mile
> T = 30 minutes * 1 hour/60 minutes = 0.5 hours
> R = D/T = 0.5 miles / 0.5 hours = 1 mile/hour
> Jack and Jill walked at an average rate of 1 mile/hour.

In real life, rarely does someone walk at exactly the same rate over an interval of 30 minutes. Calculus gives you the tools you need to investigate rates that vary over an interval. After all, Calculus is the mathematics of change.

In this activity, you will review the procedure for finding an average rate over an interval of time. Then you will investigate how to find the rate at a particular point in time. You will approach this in the same way that two of the greatest mathematicians of the seventeenth century did. Their work led to the development of modern calculus.

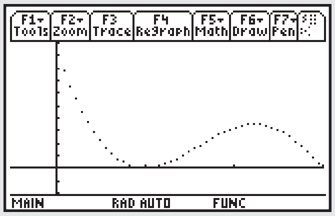
A. Real World Application

Your next-door neighbor, Phil Fitness, is a cyclist preparing for the Tour de France. He has recorded his three-hour practice ride on an odometer. You are helping him analyze his riding and have found a function that models the data.

The data can be modeled by the function $p(t) = 2t^5 - 14t^4 + 18t^3 + 46t^2 - 100t + 48$, where p is the distance from his house (in miles) and t is the number of hours since he began his practice run.

TECH-TIPS **Graphing the Problem Situation**

The only portion of the graph that applies to this problem is over the time interval 0 to 3 hours. From the Home Screen:

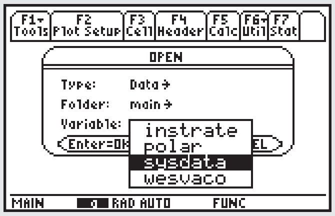


- Press [CLEAR] to clear the Entry Line on the Home Screen.
- Press [2nd] [F1], which is [F6] Clean Up. Press [2]: NewProb, then press [ENTER].
- Press [◆] [F1] for [Y=], then enter the function $p(t)$ as y1.
- To change the graphing style to dot style, highlight y1, press [2nd] [F1] Style, then press [2]: Dot.
- Press [◆] [F2] to set the viewing window to [-0.5, 3, 1] by [-10, 50, 5] xres=3.
- Press [◆] [F3] to see the graph.

Plotting Data from the Graph Screen

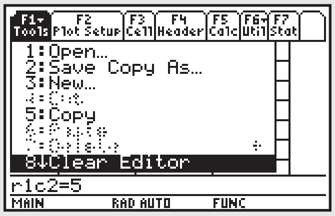
In this section we will gather data from points of interest. We will find their coordinates and use the data to create a Data Plot.

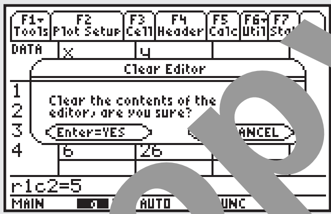
TECH-TIPS **Clearing a Data File of Previous Entries**



- Press [APPS] [6]: Data/Matrix Editor. Press [2]: Open.
- Press ⊖ to move to Folder. The symbol → indicates that you may have other folders in your calculator.
- Press ⊖ to move to Variable. The symbol → indicates that you may have other data files in the selected folder. Press ⊙ to see the list of files.
- Highlight **sysdata**, then press [ENTER].
- Press [ENTER] a second time to open the file.

- If there are any entries in the data file, press [F1] [8]: Clear Editor, then press [ENTER]. This will clear all data from the file.



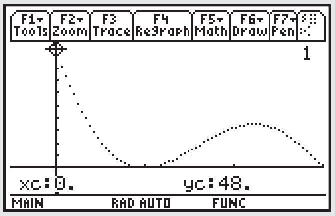


TECH-TIPS **Setting Up and Graphing a Data Plot**

Locate the point on the graph for $x = 0$.

- Press [◆] [F3] for the Graph Screen
- Press [F3] Trace, [0] [ENTER].

The cursor moves to the point (0, 48).

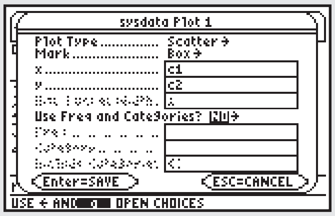


- Press [◆] [,] to place the coordinates of this point in the data file **sysdata**. The x-coordinate of the point is in column 1 and the y-coordinate in column 2.
- Repeat the same procedure for x = 1, 2, an
- Press [APPS] [6]: Data/Matrix Editor to see the entries in the data file, then [1] to open the current file, **sysdata**.

To set up a data plot:

- Press [F2] Plot Setup, then with Plot 1 highlighted, press [F1] Define.

Define the plot as shown in the screen at the right.



- Press [ENTER] to save to **sysdata**, then [ENTER] again to return to the Data Editor.

To graph Plot 1 and y1, press [◆] [F3].

1. Sketch your graph in the space provided.

2. Describe the route that Phil took. Be sure to use the correct units of measure and complete sentences. ______________________

____

____

(Hint: When $y = p = 0$, Phil is at home. What other information do you need to know in order to precisely describe his route? On the Graph Screen of your calculator, use the [F5] Math Menu to help you find the local maximum value.)

Home Screen Calculations and Interpretations

3. What was Phil's average velocity from the time that he started until the first time he arrived home? Be sure to use correct units and explain the significance of the sign. ____

____

____

4. What was his average velocity over the time interval [1, 2]? Be sure to use correct units and explain the significance of the sign.

____

____

B. Do What Newton Did

What was Phil's velocity at exactly 2 hours after he began, that is, when $t = 2$?

Unfortunately, the methods you learned in Algebra I to calculate rate, or distance divided by time, do not apply in this situation. In answering questions #3 and #4, there were 2 data points. This question asks for the velocity at one particular point.

Sir Isaac Newton discovered a method to solve this problem. As you work through the directions that follow, you will discover his method, a controversial and misunderstood topic in his day.

We are trying to find the velocity at $t = 2$. In the previous examples, we were finding the average velocity over an interval of time. Since we cannot apply this method when we have only one point, we will do the next best thing:

- Find the average rate over smaller and smaller intervals of time.
- Observe what happens to the average velocity as the intervals get very small.
- Observe what happens to the segment containing the two points on the graph of $f(x)$ that are endpoints of the interval.

1. On the Home Screen, calculate the average rate that Phil was traveling on an interval that is close to the point $(2, p(2))$. Start by calculating the following:

a. ΔD = ________________ (Enter and evaluate: y1(2.6) – y1(2).)

b. Δt = ________________ (Enter and evaluate: 2.6 – 2)

c. Calculate the ratio $\frac{\Delta D}{\Delta t}$ on the Home Screen as outlined below. Remember that the numerator of the expression for $\frac{\Delta D}{\Delta t}$ must be enclosed within parentheses $R = \frac{\Delta D}{\Delta t} =$ ____

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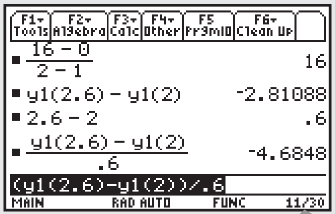
TECH-TIPS Using the History Area to Create an Entry Line

- Press CLEAR to clear the Entry Line.
- Press ⏶ ⏶ ⏶ ⏶ to move up into the history area to highlight y1(2.6) – y1(2).
- Press ENTER. The new entry will appear in the Entry Line.
- Enclose this expression within parentheses.

  2nd ◀ moves the cursor to the beginning of the line and 2nd ▶ moves the cursor to the end.

- Enter ÷ .6, then press ENTER.



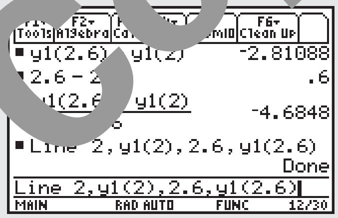
TECH-TIPS Using the Line Command

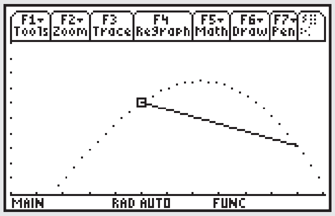
Draw the segment between the two points (2, y1(2)) and (2.6, y1(2.6)).

- From the Home Screen, change the viewing window by entering 1.5 → xmin: 2.7 → xmax: 0.1 → xscl: 10 → ymin: 20 → ymax: 1 → yscl: 4 → xres, then press ENTER.
- From the Home Screen, press CATALOG 4, which takes you to the top of the catalog listings that begin with an "L" .
- Press ⏷ as needed to highlight Line and press ENTER.

  The Line command draws a segment between two points.
  The entry Line 1, 2, 3, 4 draws a line segment between the points (1,2) and (3,4).

- Complete the command entry as shown.
- Press ENTER. Your graph should look like the one shown.





2. What is the relationship between the average rate over the interval [2, 2.6] and the segment with endpoints (2, y1(2)) and (2,.6, y1(2.6))?

____

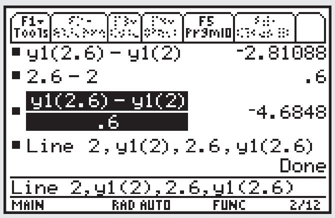
____

TECH-TIPS Editing an Entry Line

Follow the directions below to edit the expression used to calculate the average rate over the given interval.

Let $\Delta t = \Delta x = h$ $\qquad \Delta D = \Delta y = f(2+h) - f(2)$ $\qquad R = \dfrac{\Delta y}{\Delta x} = \dfrac{f(2+h)-f(2)}{h}$

- Press HOME to move to the Home Screen.
- Press CLEAR.
- Move up into the History area of the Home Screen to highlight the command as shown for finding the average rate of change.
- Press ENTER to paste the command into the Entry Line.



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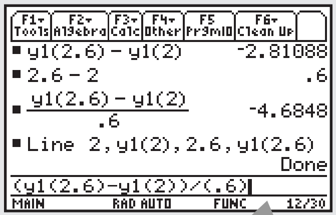
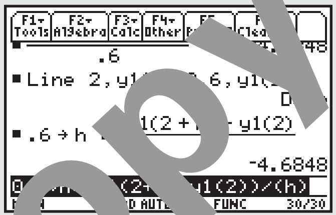
- Change 2.6 to 2+h. The letter h is [alpha] [8].
- Change 0.6 to h.
- Press [2nd] [◄] to position the cursor at the beginning of the Entry Line.
- To store 0.6 as h, press [.] [6] [STO▸] [alpha] [8].
- Press [2nd] [4], which is [:].

  This allows the calculator to execute two commands in the same line. Basically, you are telling the calculator to store 0.6 as *h*, then to calculate the average rate of change over the interval [2, 2+h].

- Press [ENTER].

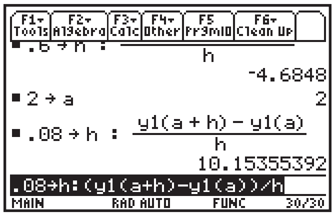
  Check the History area to see that your results are the same as shown.





3. Follow the directions below to calculate the average rate over smaller and smaller intervals of time. This gives closer and closer approximations of the rate at $x = a = 2$. Then edit the entries so that you can use the command for other *x*-values of *a* and calculate the average rate of change over the interval $[a, a + h]$.

- Press [2] [STO▸] [alpha] [=] [ENTER].
- Move into the History area to retrieve the command line you created for finding the rate of change. Press [ENTER].
- Change every *x*-value of 2 to the parameter *a*.
- Change h to 0.08. Press [ENTER].
- Substitute each of the values for $h = \Delta x$ that you see in the table below.



a. Record the results in the table below.

$\Delta r = h$	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2 + \Delta r = 2 + h$								

b. As Δx approaches ________________ from the ________________, $2 + \Delta x$ approaches ________________.

C. Historical Note

We need two points to calculate $R = \dfrac{\Delta D}{\Delta t}$ or $\dfrac{\Delta y}{\Delta x}$. When we are trying to calculate the velocity at one particular point in time, we have a problem, exactly the same problem that Sir Isaac Newton puzzled over. However, he solved the problem when he discovered his Method of Fluxion during the years 1665-66 while staying at his father's estate in the country to escape the plague in Cambridge.

Newton defined a fluxion to be "…the velocities by which every fluent is increased by its generating motion." These velocities were over an "infinitely small" interval of time. Later he wrote Principia in an attempt to explain his theory.

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"Those . . . ratios . . . are not truly the ratios of . . . quantities, but ***limits*** *toward which the ratios of quantities, . . . approach nearer . . . , but never go beyond . . ."*

In Newton's day, this explanation was not understood. As a matter of fact, Bishop Berkeley severely criticized Newton's theory in 1734. It was not until the modern concept of the limit was well established that his theory of fluxions was understood.

Translation to English: Newton's Fluxions

As a Calculus student, you have studied the concept of a limit. Translate Newton's explanation into modern English by completing the statement below.

1. In order to calculate velocity at a single point in time, "The ratio, ________________, cannot be calculated as the ratio of change in distance over change in time, but as the ________________ of the ratio as Δx approaches________________."

2. Translate the sentence into a mathematical sentence that expresses the same idea. ____

____

The concept of a limit is used to find the velocity, or the rate of change, at the instant when $x = 2$. This is called **instantaneous rate of change**. From earlier lessons, you know that a limit exists if and only if the left-hand and right-hand limits exist and are equal. In this case, you are finding the limit as Δx approaches zero.

Creating and Working With a Text Script

On the Home Screen you have created all of the calculations necessary to find $R = \frac{\Delta y}{\Delta x}$ as Δx approaches zero from the right. The same procedure can be used to investigate the left-hand limit by choosing small negative values for Δx. Instead of entering the key steps over again, you can save the calculations on the Home Screen as a Text Script. A Text Script can be easily edited and, once saved, can be used again and again to investigate other functions.

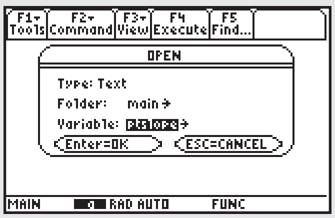
TECH-TIPS **Saving a Text Script**

- From the Home Screen, press [F1] [2]: Save Copy As…
- Press ⊖ to enter a variable name. The entry box is in alpha mode. Do NOT press alpha.
- Type the name ptslope, then press [ENTER] to save the variable name.
- Press [ENTER] a second time to save this file as a text file in the Folder you have selected.

All entries entered on the Home Screen since NewProb are now in the text file **ptslope**.

TECH-TIPS **Accessing a Text Script**

- To access this file, press [APPS] [8]: Text Editor.
- Press [2] to open a file that already exists.
- Press ⊖ then ⊳ to see the list of Variable names for the text files you have in you calculator.
- Highlight the file you would like to open, **ptslope**, and press [ENTER] to select the file.
- Press [ENTER] again to open the file.



To move to the top of a Text Script, press [◆] ⊕.
To move to the bottom of a Text Script, press [◆] ⊖.

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TECH-TIPS Clearing and/or Adding Entries in a Text Script

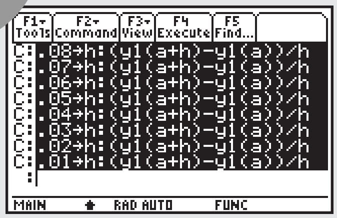
To remove any unwanted Home Screen calculation, move to that line and press [CLEAR].

The **C** at the front of a line indicates that this is a command that is to be executed. If you remove an entry, or want to use the line as a comment, remove the **C**. To remove the **C**, move the cursor to any position on the line, press [F2] [4]: Clear command.

To add an entry, move to the line above where the entry will be inserted. Move to the end of the line, press [ENTER], then type the new entry.

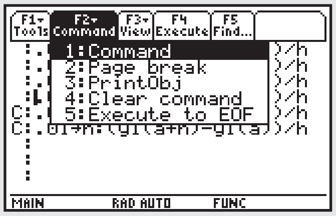
To indicate that it is a command to be executed, press [F2] [1]: Command.

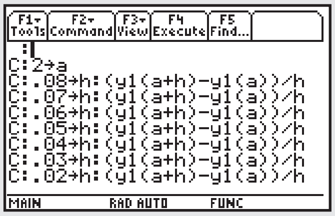
TECH-TIPS Copying and Pasting Entries in a Text Script



- Press ⊙ to move the blinking cursor to the beginning of the line $2 \to a$.

  The white up-arrow, [↑], is the **Shift** key. It is used in the same way that the Shift key is used in a computer.
- While holding down the white [↑], press the down cursor key ⊙.
  As you move down, each line will be highlighted. Move down to the last line you have executed.
- Then press [◆] [↑], which is [COPY]. Everything highlighted is now copied.
- Move to the bottom of the highlighted lines by pressing ⊙. The lines are no longer highlighted.
- Press [ENTER] to leave a blank line separating the previous commands from the lines you are about to paste.
- Press [◆] [ESC], which is [PASTE]. Your screen will look like the previous screen, with one exception: there is no **C** at the beginning of each line.
- Move up each line and press [F2] [1]: Command.





3. Complete the table below for different values of h, using the Text Script to find the rate of change of $f(x)$ as Δx approaches zero from the left.

TECH-TIPS Using a Text Script

- As you move through the lines, you will know that you are at the beginning of the new section when you get to the blank line.
- Edit each line so that the h values are negative instead of positive. Remember that you can easily move to the front or back of an entry line using [2nd], ⊙ or ⊙.
- Press [F3] View [1]: Script view to see the results of each calculation. The Text Script is on the top screen and the bottom screen is reserved for any screen dictated by the command in the Script.
- To execute each line, press [F4] Execute. The results will be shown on the bottom half of the split screen.
- Press [2nd] [APPS] to switch screens if you need to view calculations that are partially blocked.

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$\Delta r = h$	-0.08	-0.07	-0.06	-0.05	-0.04	-0.03	-0.02	-0.01
$2 + \Delta r = 2 + h$								
$R = \frac{\Delta y}{\Delta x}$								

4. Complete the following sentences.

a. As Δx approaches ________________ from the ________________, $2 + \Delta x$ approaches ________________.

b. Although we still do not have the exact rate of change at one point, we know that it is somewhere between two numbers.

________________ $< R <$ ________________

5. Conclusions: Newton's Fluxions

a. Make a conjecture for the value of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero. ____

b. Calculate the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero.

- Press ENTER to leave one or two blank lines.
- From within the Text Script, use the Catalog to obtain the **limit** command.
- Complete the entry line limit((y1(a+h) – y1(a))/h, h, 0).
- To indicate that this is a command, press F2 1: Command, F4 Execute.

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} =$ ________________

This was what Newton meant by his Method of Fluxions. The actual value of the velocity at a point cannot be calculated, but it can be found by finding the limit of the ratio of the two quantities, change in position and change in time.

D. Interpreting Your Answers

1. What unit of measure should be used with each of the following?

a. Δy ________________

b. Δx ________________

c. $\frac{\Delta y}{\Delta x}$ ________________

2. What is the instantaneous rate of change of $f(x)$ at $x = 2$? ____
Be sure to include the proper unit of measure.

3. Write the meaning of your answer in the context of Phil's practice ride. The position function is $p(t)$ and $h = \Delta t$.

Use complete sentences. ____

____

Derivatives: *Average Rate and Instantaneous Rate*

4. Explain how your answer was calculated. Write the explanation in your own words using complete sentences.

____

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____

____

5. Write a mathematical sentence that explains how the instantaneous velocity of a position function, $f(x)$, is calculated at $x = a$.

____

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E. **Graphical Interpretation of** $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

In Part B, you calculated the values of $\frac{\Delta y}{\Delta x}$ when $\Delta x \to 0$ from the right. In Part C, you calculated the value of $\frac{\Delta y}{\Delta x}$ when $\Delta x \to 0$ from the left. These values were calculated as the average rate of change, $\frac{y1(2+h)-y1(2)}{h}$, over the interval, $[2, 2 + h]$. In this section, we will consider the graphical meaning of these values.

1. Lines that intersect the graph of a function in two points are called ______________________________.

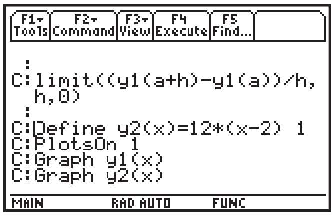
2. What is the value of $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ for $x = 2$? ________________

3. Write the equation of a linear function containing the point $(2, f(2))$ with slope equal to $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Call this line $L(x)$.

$L(x) =$ ________________

Using the Catalog to obtain the commands, type the following entries into the Text Script.

- Define $y2(x) = 12 \cdot (x-2) + 16$
- PlotsOn 1 (This turns on Plot 1.)
- Graph y1(x).
- Graph y2(x).
- Press [F2] Command [1]: Command on each new entry. Then press [F4] to execute each line.



4. Sketch the graph in the space provided.

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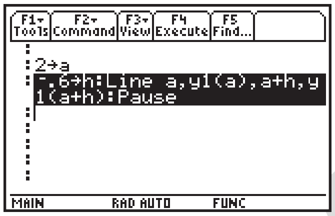
5. In Sections B and C, the rate was calculated for values of *h* close to but not equal to zero. For a graphical interpretation, use the Line command to see how changing the value of *h* affects the secant lines.

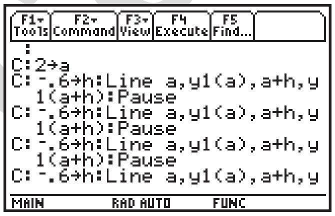


- To clear the split screen, press [F3] View [2]: Clear split.
- To move between the graph and the Text Script, press [2nd] [APPS].
- At the end of the Text Script, press [ENTER] several times to signify a section break.
- Press [2] [STO▸] [alpha] [=], which is *a*.
- Press [(-)] [.] [6] [STO▸] [alpha] [8], then [2nd] [4].
- Use the Catalog to obtain the Line and Pause commands and complete the entry with Line a, y1(a), a+h, y1(a+h):Pause.

Your screen should look like the one shown.

This command will need to be repeated for all of the *h*-values in the chart below. Use the directions that follow to repeat the command.

Copying and Pasting Commands



- Highlight the command that you just created. Hold down the shift key [↑] to help you highlight more than one line.
- Press [◆] [↑], which is [COPY].
- Press ⊖ to move to the next line.
- Press [◆] [ESC] to [PASTE] on that line.
- Repeat the last two steps nine more times.
- Press [F2] [1] on each line to signify that it is a command to be executed.

Executing the Commands: Sketching the Results

- Edit the number stored as *h* to use each of the values shown below.
- As each new segment is drawn on your calculator, sketch the segment on your graph for #4. Extend the segment in your sketch to the edges of the Graph Screen.
- The Trace command leaves you on the Graph Screen after you execute each line. To return to the Text Script, press [ENTER].

h	-0.6	-0.5	-0.3	-0.1	-0.05	0.6	0.5	0.4	0.3	0.2	0.1

Explain how the graph illustrates the concept of a limit. ____

____

____

6. Explore the relation of $L(x)$ to $f(x)$ at the point $(2, f(2))$.

- Press [2nd] [ESC] to exit the Text Script.
- If you are in Split Screen, press [2nd] [ESC] again.
- To clear the secant segments, go to the Graph Screen and press [2nd] [F1], which is [F6]. Then press [1]: ClrDraw.

a. For values of x close to 2, find all points of intersection of $L(x)$ and $f(x)$. On the Graph Screen, use [F5] Math [5] Intersection.

____

b. A line that intersects a curve in only one point is called a ________________ line.

c. The value of $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is the ________________ of the ________________ line at the point $(2, f(2))$.

Derivatives: *Average Rate and Instantaneous Rate*

Translation to Mathematics: Leibniz's Notation

Gottfried Wilhelm Leibniz served the dukes in the court of Hanover. He was the greatest inventor of mathematical symbols. We still use his notation today.

Let $y = f(x)$ be a function that models the distance traveled over a period of time. Then the ratio, $\frac{\Delta D}{\Delta t}$, becomes the ratio $\frac{\Delta y}{\Delta x}$. Leibniz converted Greek symbols into Roman symbols. Using his notation, Newton's solution for finding instantaneous velocity, or rate of change at an instant becomes:

$$\text{Instantaneous Rate} = \lim_{\Delta t \to 0} \frac{\Delta D}{\Delta t} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

The symbol $\frac{dy}{dx}$ is interpreted to mean the derivative of a function $y = f(x)$.

$\frac{dy}{dx}$ evaluated at $x = a$ gives the instantaneous rate of change of $f(x)$ when $x = a$.

It is calculated as $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

The expression $\frac{f(a+h) - f(a)}{h}$ is known as a Difference Quotient.

$\frac{dy}{dx}$, evaluated at $x = a$, also gives the slope of the tangent line at the point $(a, f(a))$.

7. Find $\frac{dy}{dx}$ at $x = a$ for each of these four functions using the Text Script created in this lesson.

- Enter the function as y1 in your Equation Editor. Turn off any other graphs.
- Graph in an appropriate viewing window.
- Use the Text Script **ptslope** to help you discover the instantaneous rate of change at a single point.

a. $p(x) = 2x^5 - 14x^4 + 18x^3 + 46x^2 - 100x + 48$ at $a = 2.4$ ________________

b. $f(x) = -0.2 \cdot (x-4) \cdot (3x-2)^3 + 5$ at $a = 2$ ________________

c. $g(x) = (x-3)^2 \cdot (5-x)$ at $a = 4$ ________________

d. $h(x) = (x-3)^{\frac{2}{3}} + 2$ at $a = 3$ ________________

Name:

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Journal Entries

Let $f(x)$ represent a position function over a period of time. Write your answers in complete sentences.

Average Rate of Change

1. What expression calculates the average rate of change over the interval $[x, x + h]$?

2. Graphically, the value of the expression in #1 is the _______ of a _______ line that contains the two points _______ and _______.

3. Sketch the graph of a curve in the first quadrant over the interval [a, b].
- Label a, b, $f(a)$, and $f(b)$ on the proper axes.
- Draw a line segment that contains the point $(a, f(a))$ with a slope equal to $\frac{\Delta y}{\Delta x}$ on the interval $[a, b]$.

Instantaneous Rate of Change

4. What expression calculates the instantaneous rate of change at one point in time?

5. Graphically, the value of the expression in #4 is the _______ of the _______ line at the point $(x, f(x))$.

6. Place $x = c$ somewhere within the interval (a, b). Label c on the x-axis.
- Locate and label the point $(c, f(c))$ on the function.
- Draw a line segment that contains the point $(c, f(c))$ with a slope equal to the $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$, where $\frac{\Delta y}{\Delta x}$ refers to the interval $[a, b]$.

7. Let $f(x) = x^2$. Calculate $\frac{\Delta y}{\Delta x}$ for $a = 2$ and $h = 0.01$, $h = -0.01$.

8. Calculate $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ for $f(x) = x^2$ and $a = 2$. Show your work.

9. Use the text script **ptslope** to find the slope of the tangent line of $f(x) = x^2$ at $a = -2$.

10. Find the slope of the tangent line of $f(x) = x^2$ at $x = -2$. Use the information that you have learned from this lesson. Show each step of your calculations.

Reference: A Concise History of Mathematics, Third Revised Edition, by Dirk J. Struik. Dover Publications, Inc., New York. Pages 110 – 111.