Name:

Integrals: *Patterns and Accumulation: Sequences and Series*

Much of the power of mathematics is in finding a pattern of how something in nature behaves. Then a mathematical model of that pattern can be used to predict behavior in the future.

In Algebra I, you first explored patterns like the one shown in the table.

x	0	1	2	3
y	5	7	9	11

This is a linear pattern that can be modeled with the equation $y = 2x + 5$.

In this lesson, you will investigate other kinds of patterns, particularly **sequences** and **series**. It is important to note that sequences are defined only for non-negative integers. Functions that are continuous on an interval are defined at every point between and including the endpoints of the interval.

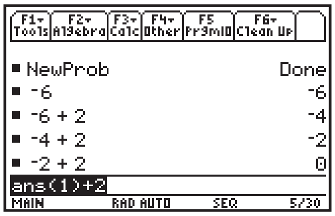
Throughout the lesson you will learn how the technology of the TI-89 can assist you in understanding the concept of accumulation. This will introduce you to the concept of a definite integral, the third big topic in calculus. Learn the basics well.

An Overview of Sequences and Series

A **sequence** is an ordered list determined by a formula, which may be stated recursively or explicitly. The items in the list are represented by u_n, where n is a non-negative integer that gives the placement of the term. For example, u_5 is the fifth term of a sequence that begins with u_1. Some sequences may begin with $n = 0$, in which case, u_5 would be the sixth term.

Recursive Formulas

A **recursive formula** gives an initial term, then the following terms are defined in relation to one or more of the preceding terms. The steps below illustrate a simple example.



- On your Home screen type [2nd] [F1] [2], which is **NewProb**.
- Press [(-)] [6] [ENTER]. Let this be the first term, $u_1 = -6$.
- Press [+] [2] [ENTER]. Each time you press [ENTER] you see the next term in the sequence.

This process can be repeated again and again. That is to say, the formula recurs each time and produces the next term in the sequence.

This sequence is called an **Arithmetic Sequence** since there is a common difference between each of its consecutive terms.

In this case, $u_n - u_{n-1} = 2$. The common difference is 2.

1. Complete the first two lines of the table below.

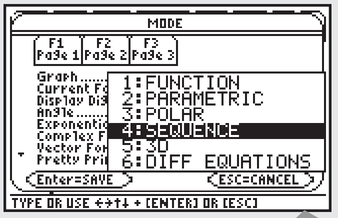
n	1						
u_n	-6						
S_n	-6						

A **series** is the sum of the terms of a sequence. For the example above, fill in the third row of the table above.

TECH-TIPS Graphing Sequences and Series

The directions below allow you to plot the points (n, u_n) and (n, S_n) on your calculator.

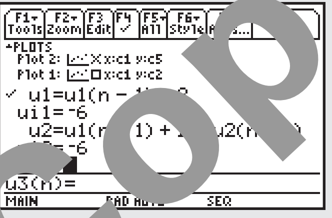
- From the Mode menu, change the Graph Mode to Sequence.



- In the Y= Editor, enter u_n and S_n as shown for $n = 1$ and $n = 2$.

Let u1 = $u_n = \begin{cases} u_{n-1} + 2, & n > 1 \\ -6, & n = 1 \end{cases}$

Let u2 = $S_n = \begin{cases} (u_{n-1} + 2) + S_{n-1}, & n > 1 \\ -6, & n = 1 \end{cases}$

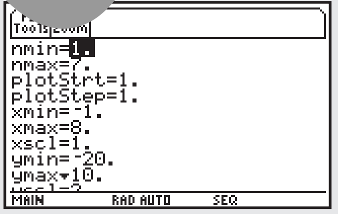


- Set the viewing window as shown.

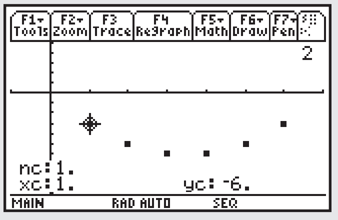
The values of nmin and nmax determine which terms are calculated. Remember that nmin must be greater than or equal to zero. Also, both of these window variables must be integers.

plotStrt =1 indicates that the first point to be plotted is for n=1.
plotStep=1 indicates that each succeeding term is plotted.

When you have a great many terms to be plotted over a large interval, you may desire to plot only every other point or every third point. This speeds up the plotting process.



- Turn off the graph of u1.
- Graph u2. Your graph should look like the one shown.



2. The value of u2(n) decreases until n = ________________. Why? ____

____

3. The value of u2(n) stays the same for n = ________________ and n = ________________. Why? ____________________________

____

4. Then the value of u2(n) increases when n = ________________. Why? ____

____

Explicit Formulas

Instead of using previous terms, an **explicit formula** defines a sequence or series in terms of a nonnegative integer n. Since the formula does not depend on previous terms, there is no need to state an initial value.

5. Find an explicit formula for the sequence, u1, above.

- Make a conjecture for the formula that explicitly defines u1, making adjustments as needed.
- Enter the explicit formula as u3 in your calculator. Use Dot Style for u1 and u2. Use Square Style for u3.
- Turn on the graphs of u1 and u3, turn off u2.

a. Trace on the graph of u3 to verify that its values match those of u1. Once you have verified that the recursive and the explicit formulas are equivalent, Write the explicit formula for the sequence. u3 = ____

b. Sketch the graph of u3 in the space provided.

6. Find an explicit formula for the series, u2, above.

- Make a conjecture for the formula that explicitly defines u2. Make adjustments as needed. Two quadratic forms that may help you determine the equation are $u_4 = a(n-h)^2 + k$ and $u_n = a(n-r_1)\cdot(n-r_2)$.
- Enter the explicit formula as u4 in your calculator. Use Square Style so that you will be able to distinguish between u2 and u4.
- Turn on the graphs of u2 and u4, turn off u1 and u3.

a. Trace on the graph of u4 to verify that its values match those of u2. Once you have verified that the recursive and the explicit formulas are equivalent, write the explicit formula for the series. u4 = ____

b. Sketch the graph of u4 in the space provided.

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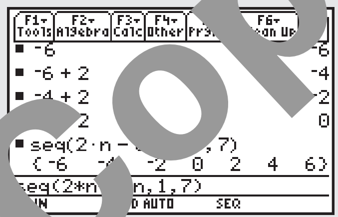
Calculating Sequences and Series

As we have seen from the graphs in the previous section, the definitions for u2 and u4 are equivalent forms. They are the accumulated sums of arithmetic sequences, hence, u2 and u4 are called arithmetic series.

- The recursive form of the arithmetic sequence is u1.
- The explicit form of the arithmetic sequence is u3.
- The recursive form of the arithmetic series is u2.
- The explicit form of the arithmetic series is u4.

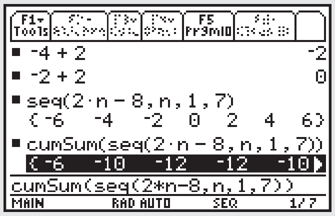
TECH-TIPS Calculating Sequences



- From the Home screen, press [2nd] [5], which is the [MATH] menu.
- Press [3] to select 3:List.
- Press [1] to select 1:seq(.
- Complete the command by entering the explicit formula, the variable, the beginning value of the variable, and the ending value of the variable. An optional parameter indicates the step size. If none is stated, the default is 1. End with a closing parenthesis and press [ENTER].
- Verify that the numbers generated by the sequence command agree with the numbers you wrote in row 2 of the table in problem #1.

7. Write the calculator syntax for the sequence command. ____________________

TECH-TIPS Calculating Series

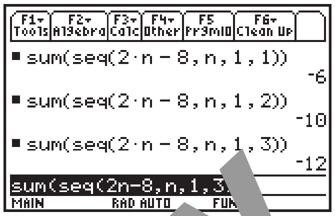


- Clear the Entry Line.
- From the Home screen, press [2nd] [5], which is the [MATH] menu.
- Press [3] to select 3:List.
- Press [7] to select 7:cumSum(.
- Press [2nd] [5] for the [MATH] menu.
- Press [3] to select 3:List.
- Press [1] to select 1:seq(.
- Complete the command by entering the explicit formula as shown.
- Verify that the numbers generated by the cumulative summation command agree with the numbers you wrote for S_n in the table. This command results in a list of numbers each of which is an accumulation of the previous terms of the sequence.

8. Write the calculator syntax for the cumulative summation command. ____________________

Essentially, this command says, "Accumulate the sum of the terms of the sequence, $2n - 8$, with respect to n, as n takes on the integral values from 1 to 7."

At times, you may be interested in a single summation of certain terms. In this case, you would use the **sum** command, 6:sum(from the List command. It returns one result, which is the sum of specified terms in the sequence. To verify that S_3 = -12 from your table in problem 1, enter the last command as shown. The last two digits indicate that you want to sum the first through third terms.



There is a more efficient way to write the sum command using **sigma notation**. Sigma notation tells the calculator to add the terms of a sequence of numbers. The sum of those numbers is the result.

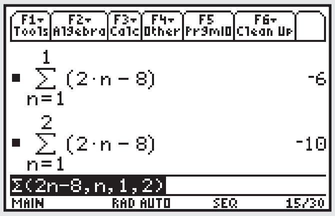
TECH-TIPS Using Sigma Notation

- From the Home Screen, press F3 Calculus, then [4] to select 4: Σ(sum.
- Enter the explicit expression that defines the sequence, $2n - 8$.
- Enter the second argument, n.

  This is the parameter that varies, called the **index variable**.

- To find S_2, enter the next two arguments, 1 and 2, whi[illegible] ll the [illegible] ulator the initial and final terms of the sum, then press [ENTER].

  The result is called a **partial sum**.



9. Calculate each partial sum S_1 throug[illegible] us[illegible] e sigma notation command of the TI-89. Write the results and verify that they agree with the numbe[illegible] you [illegible] te in r[illegible] 3 of the table in problem #1.

S_1 = ________________ [illegible] = ________________ S_3 = ________________

S_4 = ________________ S_5 = ________________ S_6 = ________________

10. Calcul[illegible] $\sum$ [illegible] $k+3)$ by hand. ________________ Show your work.

k	1	2	3	4
u_k				
S_k				

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Assessment

1. Write the set of numbers generated by these sequence commands. Then check your answers using the TI-89.

a. $seq\left(x^2, x, 1, 7\right)$ ____

b. $u_i = \begin{cases} 4, i = 0 \\ u_{i-1} \cdot 2, 0 < i \le 5 \end{cases}$ ____

c. $u_n = 2n^2 - n, 1 \le n \le 4$ ____

2. State which of expressions in question 1 is a recursive formula ________________; an explicit formula ________________; and a calculator syntax ________________.

3. Calculate the values of the following using pencil and paper, then check with a calculator. Show your work.

a. $\sum_{i=3}^{6} 4i$ ________________

b. $sum\left(seq\left(n^2 - 1, n, 1, 5\right)\right)$ ________________

c. Find S_4, for $S_n = \begin{cases} 2n + 10, n = 0 \\ 2n + 10 + S_{n-1} \end{cases}$ ________________

4. Given: $u_n = 2n - 10$. Fill out the table below to find $\sum_{n=0}^{5} u_n \cdot \frac{1}{2}$ =

n						
u_n						
$u_n \cdot \frac{1}{2}$						
$S_n = \sum_{k=1}^{n} u_k \cdot \frac{1}{2}$						

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The following are examples of **infinite series**. See if you can determine the number or function represented by the following expressions. Use the hints that are given for each example. Show your work.

5. $\sum_{n=1}^{\infty} \frac{3}{10^n}$ = ________________

(HINT: Make a table like the one above for n, u_n, and S_n.)

6. $\sum_{n=1}^{\infty} \sin(\pi \cdot n)$ = ________________

(HINT: Graph the expression for u_n in function mode. Trace to integral values of x, then add.)

7. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ = ________________

(HINT: In function mode, graph y1 = S_1. Then add the next term in the sequence and graph. Each time that you add another term, you will have a polynomial of higher degree. When you think you know the function that is approximated by this series, enter it into y2 and check it against the graph of y1. Keep adding terms of the series to get a better approximation.)

n	1	2	3	4	5	6	7	8	9	10
u_n										