Name:

CHAPTER 5

Lesson 22

Connections: *THE Mean Value Theorem*

Introduction

The concept of "average" or "mean" has very important applications in the real world. For example: sports, economics, business, statistics, and weather. In Calculus, this concept is required in the Fundamental Theorem of Calculus.

You have studied the Mean Value Theorem for Derivatives.

The Mean Value Theorem for Derivatives

says that if a function is
continuous at every point of a closed interval [*a*, *b*]
and differentiable at every point of the open interval (*a*, *b*),
then there is at least one number *c* between *a* and *b* for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

You have also studied the Mean Value Theorem for Definite Integrals.

The Mean Value Theorem for Definite Integrals

says that if a function, *f*, is
continuous at every point on the closed interval [*a*, *b*],
then at some point *c* in [*a*, *b*],

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

In this lesson, you will explore the connection between these two theorems.

The Use of "Average" in Mathematics

In the lower grades, you were taught to average a list of numbers. At some point you learned that another term used for "average" is "mean". In word problems, you learned the formula $D = R \cdot T$, where *R* represents an average rate over a time interval.

Example 1:
Susie travels for 3 hours. She traveled 150 miles. What was her average speed during that time?

Solution:

$$D = R \cdot T$$

$$R = \frac{D}{T} = \frac{150 miles}{3 hours} = 50\frac{miles}{hour}$$

Notice how the units are used as a check for the method chosen. The answer, 50 miles per hour, is what you would expect when finding a rate. Often, knowing the units in a problem will help you determine the method to be used for the solution.

Units are emphasized on AP Calculus Exams now more than ever. They should always be included in your answers, especially if you are asked to interpret the meaning of an answer.

Check:

$$D = 50\frac{miles}{hour} \cdot 3 hours = 150 miles$$

When you were introduced to the concept of a "function", you were first taught linear functions. Using a constant rate over some interval of time, you learned that distance is a linear function of time, $D = f(t) = 50t$. If there is also some initial position, then the form becomes $f(t) = y = y_0 + 50t$, as scientists prefer; or $y = mx + b$ as mathematicians prefer.

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The Use of "Average" in Calculus

In the real world, it would be unreasonable to expect to travel at a constant rate for any length of time. Until now, you have not had the tools to deal with such a situation precisely. It is only in Calculus class that you have been introduced to the concept of rates that vary.

Consider a European train. It starts at a station and travels slowly at first while it is close to the city, then faster as it gets further away, then very fast in the open countryside, then slower as it approaches the next station.

Example 2:
The following function illustrates the description above.
Let $f(x) = x^3 \cdot (\sin(x))^2 + 15 \cdot x^2$,

x = time in hours
$f(x) = y$ = distance traveled in miles.

A. The Position Function

1. Enter this function in your calculator as y1.

- Find an appropriate viewing window for the first six hours of the trip.
- Write the function, then sketch the graph in the space provided.
- Be sure to label the axes indicating what each axis represents, including units.
- Indicate the viewing window used.

2. Now consider only the part of the trip from $t = 1$ to $t = 6$.

- On the graph you sketched in #1, label the point $P(1, f(1))$ and $Q(6, f(6))$.
- On the Home Screen, use the formula $D = R \cdot T$ to calculate the average velocity of the train over that time period.
- Store this value as m.
- Justify your answer in the space provided. Be sure to include units in your answer.

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3. Give a geometrical interpretation to the answer to #2. Add feature(s) on your graph in #1 to illustrate your answer. ______________

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4. On your calculator, enter the equation of the line PQ as y2 = $m \cdot (x - 1) + y1(1)$.

- Graph y1 and y2 on your calculator to verify that it matches the sketch you have drawn.
- Write the equation for PQ substituting the actual values for m and y1(1).

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B. The Velocity Function

Since the value *m* is the value of a constant velocity, we will investigate the velocity function, where the velocity varies.

On the Home Screen, derive the velocity function, $f'(x)$.

- Use [F3] [1] Differentiate or [2nd] [8] which is *d*(and enter y1(x),x).
- Copy and paste this expression into the Y= Editor as y3.

TECH-TIPS **Copying and Pasting**

- Highlight the expression that you wish to copy.
- Press [◆] [↑] which is Copy.
- Move to the location where you wish to paste the copied expression.
- Press [◆] [ESC] which is paste.

5. Write the equation for y3 = $f'(x) =$ ____

- Turn off y1 and y2. Turn on y3 and graph in an appropriate window.
- Sketch the graph in the space provided.
- Be sure to label the axes indicating what each axis represents, including units.
- Indicate the viewing window.

6. On the Home Screen, calculate the Average Value of $f'(x)$ over the interval [1, 6].

- Show your calculation in proper Calculus notation. ____
- Store this value as *v*.

7. Let y4(x) = *v*.

- Add the graph of y4(x) to your sketch in #5.
- On the interval [1, 6], how many points of intersection are there? ____________________

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C. Two Mean Theorems

Let's take a closer look at the two mean value theorems as they apply to the position function $f(x)$ and the velocity function $f'(x)$.

The Mean Value Theorem for Definite Integrals

says that if a function, *f*, is continuous at every point on the closed interval [*a*, *b*], then at some point *c* in [*a*, *b*],

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

In other words, "the average value happens somewhere", and it "happens" within the defined interval. In this case, the average value of the velocity function happens at some point in time, c, $1 \le c \le 6$ hours. Stated in mathematical terms,

$$f'(c) = \frac{1}{6-1}\int_1^6 f'(x)dx.$$

8. In order to answer the question, "When was the train traveling at its average velocity?", we need to find the *x*-coordinates where the graphs of velocity and average velocity intersect.

The function $f'(x)$ takes on the average value found in #6 twice.

- From the Graph Screen, find the first intersection of y3 = $f'(x)$ and y4 = v = average value of $f'(x)$. The coordinates of that intersection point are stored temporarily as (xc, yc).
- Store xc as *c*.
- Find the second intersection point and store its *x*-coordinate as *d*. Label *c* and *d* on your sketch.

- *c* = ____________________ *d* = ____________________

- In a complete sentence, explain what these values mean, including units. ____

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In parts A and B of this lesson, you investigated the graphs of y3(x) = $f'(x)$ and y4(x) = *v* = average value of $f'(x)$. You have also found the values of *c* and *d* where the velocity is equal to the average velocity.

Now, you will investigate the graphs of y1(x) = $f(x)$ and y2, which is the line connecting the endpoints *P*(1, f(1)) and *Q*(6, f(6)). As you work through this lesson, the goal is to see how all the graphs are related and how the two mean value theorems are connected.

9. Investigate the graphs of y1(x) = $f(x)$ and y2(x) = $m(x - 1) + f(1)$.

- Turn off y3 and y4. Turn on y1 and y2.
- Refer to #1 for window settings.
- Graph the two functions. Sketch your graphs in the space provided.

a. Explain what the graph of y2 represents. ____

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- Press F5 Math and select A: Tangent. The calculator will ask for the x-coordinate of the point where the tangent line will be drawn.

b. Type c first, and record the equation. ______________________________

c. Repeat for $x = d$. Record the equation ______________________________

d. Sketch these lines on your graph on page 147.

You will not be able to trace on these lines since they are drawn and not defined as a function.

e. How do these tangent lines relate to the line PQ? ______________________________

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Journal Entries

1. Enter the following in your journal to help you understand the answers to two major questions.

I. What have you learned about the relationship between the graphs of the functions
$y1(x) = f(x)$,
$y2(x) = m(x - 1) + f(1)$,
$y3(x) = f'(x)$,
and $y4(x) = v$?

II. How are the two mean value theorems related?

a. Explain how y1(x) and y3(x) are related.

b. Explain what"Average Velocity" means for $y1(x) = f(x)$.

c. Explain what"Average Velocity" means for $y3(x) = f'(x)$.

d. Explain the role of c and d in relation to $y1(x) = f(x)$.

e. Explain the role of c and d in relation to $y3(x) = f'(x)$.

f. How was the average velocity calculated for the position function $f(x)$ over the interval [1, 6]?

g. How was the average velocity calculated for the velocity function $f'(x)$ over the interval [1, 6]?

Connections: *THE Mean Value Theorem*

Conclusion

The Mean Value Theorem for Derivatives

says that if a function is
continuous at every point of a closed interval [*a*, *b*]
and differentiable at every point of the open interval (*a*, *b*),
then there is at least one number *c* between *a* and *b* for which

$$\boldsymbol{f'(c) = \frac{f(b) - f(a)}{b - a}}.$$

In other words, "it happens". Somewhere between $x = a$ and $x = b$, the mean, or average, rate of change "happens".

- At some point, the instantaneous rate of change, $f'(c)$, is equal to the average rate of change, $\frac{f(b) - f(a)}{b - a}$.
- Geometrically, the slope of the secant line *PQ* is equal to the slope of the tangent line at (c, $f(c)$). In other words, the secant line over the interval [*a*, *b*] is parallel to the tangent line when $x = c$, where *c* is in the interval (*a*, *b*).

The Mean Value Theorem for Definite Integrals

says that if a function, *f*, is continuous at every point on the closed interval [*a*, *b*],
then at some point *c* in [*a*, *b*],

$$\boldsymbol{f(c) = \frac{1}{b - a} \int_a^b f(x)\,dx.}$$

In other words, "the average value of the function happens somewhere", and it "happens" within the interval [*a*, *b*].

If the function is itself a rate function, velocity for example, then the average value of that function can be calculated as:

$$f'(c) = \frac{1}{(b - a)} \int_a^b f'(x)\,dx$$

10. Write both of the mean value theorems as one theorem, THE Mean Value Theorem. Write this theorem in your journal.___________

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11. Do you need to state that $f(x)$ is differentiable on (*a*, *b*) in #10? ____