Name:

Infinite Series: *Geometric Series*

Some people are just smarter than others; I have met some of them. You may be using a text that some of them have written. Did you hear about the 9 year old **Karl Friedrich Gauss** (1777-1855) who was asked by his teacher to add the integers from 1 to 100? In no time at all he wrote the answer down and sat there looking bored. I imagine that he was bored in class most of the time.

His answer was correct, of course. What he had done was to realize that if he added all of the numbers to each other twice he would get twice their sum. Then he could divide that number by 2 to arrive at the answer. The secret lay in knowing how to pair the numbers.

$S_1 = 1 + 2 + 3 + ... + 98 + 99 + 100$, while $S_2 = 100 + 99 + 98 + ... + 3 + 2 + 1$.

Thus $S_1 + S_2 = (1+100) + (2 + 99) + (3 + 98) ... + (98 + 3) + (99+ 2) + (100 + 1) = 100$ pairs of 101, or 100 (101) and since 2 times the sum is equal to 100 (101), $S = 50(101) = 5{,}050$. Elementary, my dear Watson.

Today, we use his formula for the sum of any arithmetic sequence $S_n = \dfrac{n(a_1 + a_n)}{2}$. Like I said, some people are just smarter than others.

Enjoy this, our last chapter, as we dare to explore infinity. The ancient Greeks would not. **Rene Descartes** thought that it was foolish for finite man to ask questions dealing with infinity. So we will turn to those who were unafraid, especially the father of calculus himself, **Sir Isaac Newton**.

Hopefully, this chapter will bring your study of calculus to a conclusion that helps you to relate the parts to the whole. As you will see, this is where it all began.

Questions to Explore:

1. What makes an infinite series of numbers converge to a single value?
2. What are some infinite series that represent some important numbers?
3. What makes an infinite series of expressions converge to a single function?
4. What are some infinite series that represent some important functions?
5. How can we use series that we know to derive other functions?
6. What connection is there to the improper integral $\int_{x=1}^{\infty} \left(\frac{1}{x^p}\right) dx$?
7. How can we create a series to represent a function?

Calculator Setup

- [MODE]: Graph-**Sequence**: Display Digits-**Float 6**: [F2] **Page 2**: Exact/Approx-**Auto**: [ENTER]
- [◆] [F2] for [WINDOW]: n [1, 20]; plotStart = 1; plotStep = 1; X [-1, 20, 2]; Y [-3, 12, 2]
- [HOME] [CLEAR]: [2nd] [F1] for [F6] **CleanUp**: [2] **New Problem**: [ENTER]

How can an infinite sum of numbers converge to a finite number?

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A. HOME Screen Exploration: Sequences

1. Write the answer you get after each calculation, and the equivalent expression in terms of 3, 0.7 and n.

a. 3 . 0 ENTER $n = 1$ ____________________ = ____________________

b. × 0 . 7 ENTER $n = 2$ ____________________ = ____________________

c. ENTER $n = 3$ ____________________ = ____________________

d. ENTER $n = 4$ ____________________ = ____________________

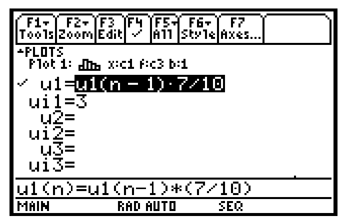
e. ENTER $n = 5$ ____________________ = ____________________

f. ENTER $n = 6$ ____________________ = ____________________

In textbooks, sequences are often defined **recursively** with one or more initial terms being specified. Then the nth term a_n is defined in relation to one or more of the preceding terms. In this case, each succeeding term can be found by multiplying the previous term by seven-tenths.

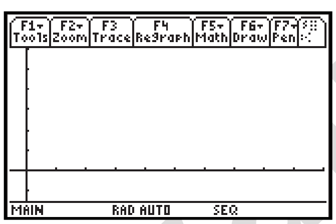
In your calculator when in **Sequence Mode**, you may define sequences **explicitly** or **recursively** as u in terms of n. The following sequence is shown recursively in terms of a_n and in calculator syntax in terms of u1(n). The term "ui1" means the initial term of the sequence u1.

2. $a_n = \begin{cases} a_{(n-1)} \cdot (0.7), & n > 1 \\ 3, & n = 1 \end{cases}$ $\quad u1(n) = \begin{cases} u1(n-1) \cdot (0.7), & n > 1 \\ 3, & n = 1 \end{cases}$



- Enter the sequence as u1 in the [Y=] **Editor** as shown.
- 2nd F1 for [F6] **Graph Style**: 3 **Square**.

a. ◆ F3 for [GRAPH]. Sketch the plot in the screen provided below.
F3 **Trace**. Label the point (x, y) = (10, u1(10)).



b. Find an **explicit formula** for the terms of the **sequence**.
That means to find a formula that is based on the number n of each term instead of the previous term a_{n-1} as it was in u1(n).
- Let u2(n) equal the explicitly defined sequence, where u2(n) = a_n = ______________.
- ◆ F3 for [GRAPH]: F3 **Trace** to see that the graphs of u1(n) and u2(n) are the same.

c. On the HOME Screen, evaluate to be certain that you have entered the equations correctly.

- u1(2) = u2(2): ______________.
- u1(10) = u2(10): ______________.

d. Display the first 4 terms of the sequence on the HOME Screen. Follow the TECH-TIP below:

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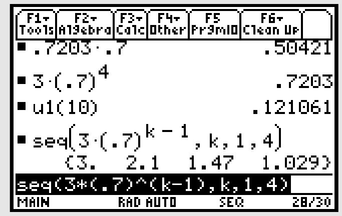
TECH-TIP [HOME] **Screen:** [MATH] **Menu:** [3] **List:** [1] **Sequence** *page 365*

seq(3 (0.7)$^{(k-1)}$, k, 1, 4)

The **sequence** command can be found in two places on the [HOME] Screen:

1. [2nd] [5] **Math Menu**: [3] List: [1] **seq(** : [ENTER]
2. [CATALOG]: [3] for the top of the **"S"** list, then ⊝ until the indicator is beside **seq(:** [ENTER].



seq(expression, variable, low value of the variable, high value of the variable [, step]).

The step size is optional, with a default of 1. Increasing the step size to 2, for example, increases the plotting speed because it only plots every other point.

B. The Accumulation of a Sequence is a Series

In this section, you will learn several ways to calculate and graph series on your calculator.

A **series** is the sum of the terms of a sequence.

3. Consider the sequence: $a_n = \begin{cases} a_{(n-1)} \cdot (0.7), & n > 1 \\ 3, & n = 1 \end{cases}$

a. Let $S = a_1 + a_2 + a_3 + a_4 + \ldots + a_n + \ldots =$ ____

- Substitute values for n into the first 4 terms in the infinite sum of the sequence above.
- Do not evaluate the terms.
- Then write the n^{th} term of the series.

b. S_n is a **partial sum** where only n terms of the series are used.

- Enter the following definitions for S_n in the [Y=] Editor of your calculator.
- After you enter each equation, [GRAPH] and [F3] **Trace** on the plot.

i. Series can be written as the **sum of a sequence**.

- Let u3(n) = sum(seq (3(0.7)$^{k-1}$, k, 1, n)) Use [2nd] [F1] for [F6] **Graph Style**: [2] **Dot**
- Press [2nd] [5] for **Math Menu**: [3] List.
- Select [6] **sum(** for summation. Select [1] **seq(** for sequence.

ii. Series can also be written **recursively**: $S_n = \begin{cases} S_{n-1} + a_n, & n > 1 \\ a_1, & n = 1 \end{cases}$

Let u4(n) represent the series Sn which is the accumulation of the first $n - 1$ terms of the sequence, beginning with 3 when $n = 1$, ending with the n^{th} term $a_n = 3.0 * (0.7)^{(n-1)}$.

Let $u4(n) = \begin{cases} u4(n-1) + 3 \cdot (0.7)^{(n-1)}, & n > 1 \\ 3, & n = 1 \end{cases}$ Use [2nd] [F1] for [F6] **Graph Style**: [3] **Square**

iii. Series can be written using **sigma notation**:

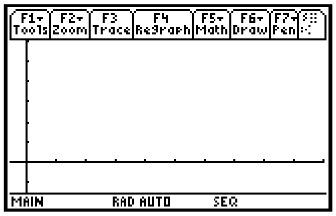
$$S_n = a_1 + a_2 + a_3 + \ldots a_{n-1} + a_n = \sum_{k=1}^{n-1} a_k + a_n = \sum_{k=1}^{n} a_k$$

Let u5(n) = $\sum_{k=1}^{n} \left[3 \cdot (0.7)^{k-1} \right]$ Use [2nd] [F1] for [F6] **Graph Style**: [3] **Square**

- In the Math Menu: Press [2nd] [5] Math Menu: [alpha] [(] for **B:Calculus**: [4] $\sum$(sum.
  $\sum$(expression, variable, low, high) = $\sum(3 * (0.7)^{(k-1)}, k, 1, n)$

c. Sketch the graphs of a_n and S_n in the screen below.

- [◆] [F3] for [GRAPH]: [F3] **Trace** to $n = 10$ on the graph of S_n. Mark that point on the graph.
- [◆] [(-)] to record the *x*- and *y*-coordinates on the [HOME] Screen.



d. Write the partial sum of the series S_{10} using Sigma notation then in expanded form.

S_{10} = ____

e. Check your answer to #3d when $n = 10$ on the [HOME] Screen.

- [HOME] [F3]Calculus: [4] $\sum($*sum* :
- $\sum(3\cdot(0.7)^\wedge(k-1), k, 1, 10) = \sum_{k=1}^{10}\left[3\cdot(0.7)^{(k-1)}\right] = 9.71752$

4. Do you think that there is a finite limit to this infinite summation? Why or why not? ______________________________

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C. Geometric Series

5. This series has the explicit form ______________________ and is an example of a(n) ________________________ series.

Each term has a common ____________________ r = ___________.

6. Find a formula for the sum of the original geometric series S1. Follow the steps below.

a. Multiply each term of the series S1 by the common ratio r to derive a new series S2.

b. Find the difference S1 − S2. Write an equation, and then solve for S1. Show your work.

c. The initial value of the series is a_1 = _____, the common ratio is r = _______, and the sum of the first n terms of the series is $S1_n$ = ____________________________.

d. As n approaches infinity, $\left[\frac{7}{10}\right]^n \to$ ____ and $\frac{3\cdot(1-[\qquad\qquad])}{(1-[\qquad])} \to$ ______________________________.

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D. Conclusions

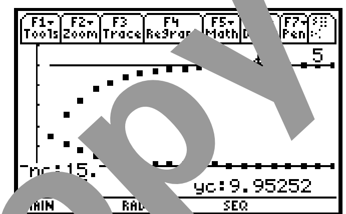
7. An infinite geometric series $\sum_{n=1}^{\infty} a_1 \cdot r^{n-1}$ has an initial value __________ and a common ratio __________.

a. The sum of the first n terms of the finite geometric series is S_n = ______________________________.

b. The sum of the infinite geometric series converges to S = _________________ when $|\, r \,|$ __________.

8. Let u6(n) = the value of convergence in #6d.

- Use [2nd] [F1] for [F6] **Graph Style**: [1] **Line** for u6.
- Graph u1, u5, and u6.
- Your graph should look like the screen at the right.
- Trace carefully on the graphs to compare values.



9. Explain why the terms of the sequence $u1$ decrease. Make connections between the recursive formula

$$u_1(n) = \begin{cases} u1(n-1)\cdot \dfrac{7}{10}, & n > 1 \\ 3, & n = 1 \end{cases}$$

and the explicit formula $a_n = 3 \cdot (0.7)^{(n-1)}$. ______________________________

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10. Do the values of u5 = $\sum_{k=1}^{n} a \cdot r^{n-1}$ ever reach the value of u6 = $\dfrac{a_1}{(1-r)}$? Use #8 to justify your answer. ____________________

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❖ ***Journal Entries***

Write the following definitions in your journal.

Infinite Series

An infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \dots + a_n + \dots, \text{ or } \sum_{k=1}^{\infty} a_k .$$

The numbers a_1, a_2, ... are the terms of the series;

a_n is the ***n*****th term**.

Infinite Series: *Geometric Series*

Partial Sum

A finite sum of the form $s_n = \sum_{k=1}^{n} a_k$

If the sequence of partial sums $s_1, s_2, s_3 \ldots s_n \ldots$ has a limit S as n approaches infinity, we say the series **converges** to the sum S, and we write:

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k = S$$

Otherwise, the series **diverges**.

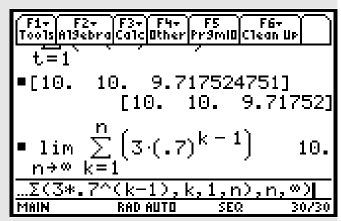
Convergence of an Increasing, Bounded Sequence

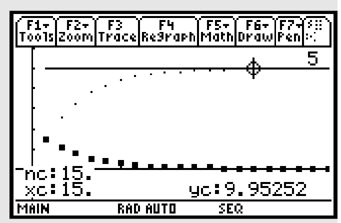
If a sequence S_n is increasing and bounded above, then $\lim_{n \to \infty} (S_n)$ exists.

If $\lim_{n \to \infty} S_n = S$, then the series $\sum_{n=1}^{\infty} a_n$ **converges** to the sum S. Then by definition $\sum_{n=1}^{\infty} a_n = S$.

If $\lim_{n \to \infty} S_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ **diverges**.

2. What does this theorem say about $S = \sum_{k=1}^{\infty} 3 \cdot (0.7)^{(k-1)}$?





We have explored the series $\sum_{k=1}^{\infty} 3 \cdot (0.7)^{(k-1)}$ in this lesson. It is an example of a geometric series, which means that each term is a multiple of the previous term. The terms of a **geometric series** $a \cdot r^{(k-1)}$ are increasing. If the common ratio r is $-1 < r < 1$, then the series is also bounded above. Therefore, the limit of the partial sum S_n as n approaches infinity exists and the series converges to a finite sum.

Geometric Series

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} a \cdot r^{n-1}$$

If $|r| < 1$, $\sum_{n=1}^{\infty} a \cdot r^{n-1}$ converges to the sum $\frac{a}{1-r}$ over the interval $-1 < r < 1$.

If $|r| \geq 1$, $\sum_{n=1}^{\infty} a \cdot r^{n-1}$ diverges.

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3. What do we mean when we write $\sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{(1-r)}$, when $|r| < 1$?

4. If we know that $\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \ldots + a \cdot r^{(n-1)} + a \cdot r^n \ldots$, then how could we write $c_0 + c_1 \cdot r + c_2 \cdot r^2 + c_3 \cdot r^3 + \ldots + c_n \cdot r^n + \ldots$? _______________

5. Since $\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \ldots + a \cdot r^{(n-1)} + \ldots = \frac{a}{1-r}$, then $\sum_{n=0}^{\infty} c \cdot x^n =$ ___________

In #6–9, by using the variable x as the common ratio in the sequence of terms, you have created something that **resembles a polynomial**. Answer the following questions **yes** if you agree, or **no** if you disagree, that the properties are the same for polynomials as for the infinite series below.

- If your answer is **no**, explain how they are different.
- Change [MODE]: Graph-**Function**
- [◆] [F1] for [Y=]: [F1] Tools: [8] **Clear Functions**: [ENTER]
- Let $y1(x) = \frac{1}{1-x}$ = the sum of the infinite series with $c = 1$ and $r = x$.
- Define $y2(x)$ as a series using sigma notation so that you can easily add an[illegible] terms.
  - You will need to add more terms before you write your final answer.
  - Let $y2(x) = \sum_{k=0}^{4} x^k$
  - **Catalog:** Press [CATALOG] [T] [⏶] [⏶] [ENTER] $\sum$ (sum
  - Calculator syntax: $\sum\left(x\wedge k, k, 0, 4\right)$ Any dummy variable will do: k, n, t, etc.
  - For $y2(x)$ use [2nd] [F1] for [F6]: Graph St[illegible]: [6] [illegible]
- Graph in the [WINDOW]: X [-1.975, 1.9[illegible]]; Y [illegible] 4, 1]; xres = 1.

$$\sum_{n=0}^{\infty} c \cdot x^n = c + c \cdot x + c \cdot x^2 + c \cdot x^3 + \ldots + c \cdot x^n + \ldots = \frac{c}{1-x}, \ |x| < 1$$

6. _____Polynomials are a sum of coefficients multiplied by powers of x.

7. _____Polynomials are continuous over the set of all real numbers.

8. _____Polynomials have finite degree.

9. _____Polynomials are either even or odd functions.